Final Exam Probability Theory (WIKR-06)

20 June 2019, 09.00-12.00

- Every exercise needs to be handed in on separate sheets, which will be collected in separate piles.
- Write your name and student number **on every sheet**.
- It is absolutely not allowed to use calculators, phones, the book, notes or any other aids.
- Always give a short proof of your answer or a calculation to justify it, or clearly state the facts from the lecture notes you are using (unless it is stated explicitly in the question that this is not needed.).
- **NOTA BENE:** using separate sheets for the different exercises and writing your name and student number on all sheets is worth 10 out of the 100 points.

Exercise 1, (a:10, b:10, c:10 pts).
A hen lays X eggs where $X \sim \mathrm{Poi}(\mu)$ is Poisson with mean μ. Each egg hatches with probability p, independently of the other eggs, yielding Y chicks.

(a) Give the joint pmf of (X, Y). (The correct answer suffices.)

(b) Show that $Y \sim \mathrm{Poi}(\mu p)$.

(c) Show that the correlation coefficient of X and Y satisfies $\rho_{X,Y} = \sqrt{p}$.

Exercise 2, (a:6, b:6, c:6, d:6, e:6 pts).
The joint pdf of (X, Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} c \cdot e^{-y} & \text{if } 0 < x < y; \\ 0 & \text{otherwise.} \end{cases}$$

Here $c > 0$ is a constant to be determined

(a) Are X, Y independent? (Justify your answer.)

(b) Determine c.

(c) Determine f_X.

(d) Determine f_Y.

(e) Determine f_{X+Y}.

Exercise 3, (a: 15, b: 15 pts)
There has been an election for the student council and there were only two candidates, let us call them A and B. The voters have cast their ballots into a sealed box. Suppose that there were a votes for A and b votes for B.

The committee in charge of counting the votes takes out the ballots one by one. The order in which they take out the ballots is completely random, because they have shaken the box really well and they don't look inside while they take the ballots out.

(a) Show that

$$\mathbb{P}(\text{ the last ballot they take out of the box is a vote for } A\ ) = \frac{a}{a+b}.$$

The committee have a blackboard at their disposal, which they've divided into two parts, the left half and the right half, separated by a vertical line. If there is a vote for A the committee make a "tally" mark on the left board and if there is a vote for B they make a "tally" mark on the right side.

(Tallies on a black board.)

We are interested in the probability that at all times during the counting process there are more tallies on the left than on the right side of the board. That is, the probability that, for all $1 \leq i \leq a+b$, among the first i votes there strictly are more in favour of A than in favour of B.

(b) Show that

$$\mathbb{P}(A \text{ is ahead during the entire counting process}) = \begin{cases} \frac{a-b}{a+b} & \text{if } a \geq b, \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: Use induction on a **and** *b with induction hypothesis "for all (a', b') with $a' \leq a, b' \leq b$ and $(a, b) \neq (a', b')$ the statement holds", and use part (a).)*